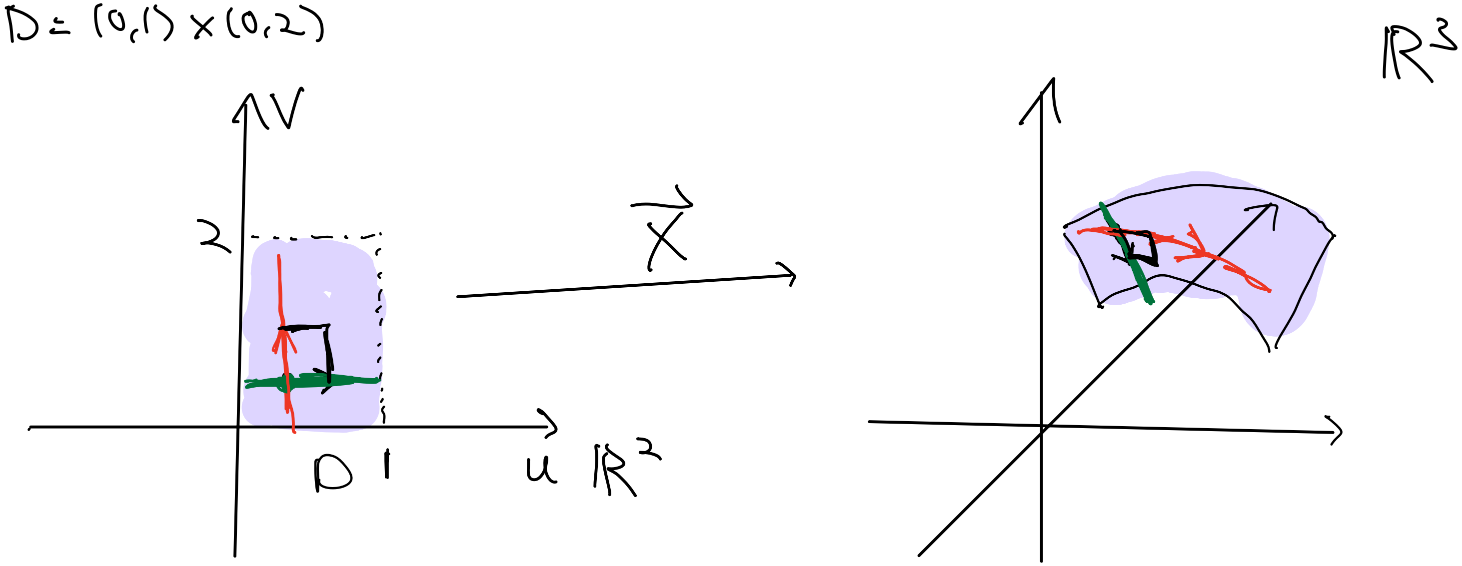
3 Surfaces

3.1 Regular parametrized surfaces

Definition 3.1.1 (Regular parametrized surface). *A* **regular parametrized surface** *is a differentiable function* $\mathbf{x} : D \to \mathbb{R}^3$*, where* $D \subset \mathbb{R}^2$ *is an open connected subset, such that* $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$*, for any* $(u, v) \in D \subset \mathbb{R}^2$*. The image* $S = \mathbf{x}(D) \subset \mathbb{R}^3$ *is called a* **regular surface**.

eg $D = (0,1) \times (0,2)$

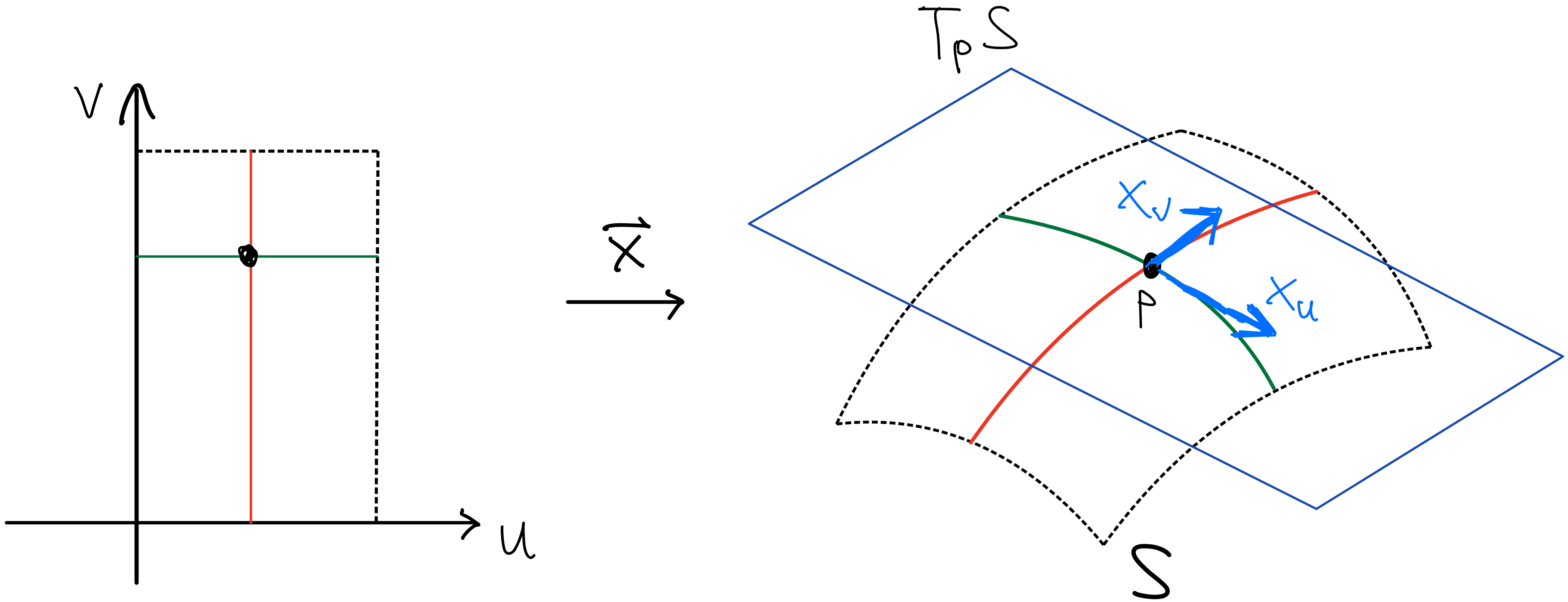


① For $\mathbf{x}(u, v)$, we denote $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$ to be the partial derivatives

② $X_u \times X_v \neq 0 \Rightarrow X_u, X_v$ are non-zero and not in the same/opposite directions

Definition 3.1.2 (Tangent space)**.** *Let S be a regular surface with parametrization $\mathbf{x}(u,v)$. The* **tangent space** *of S at $p = \mathbf{x}(u,v)$ is*

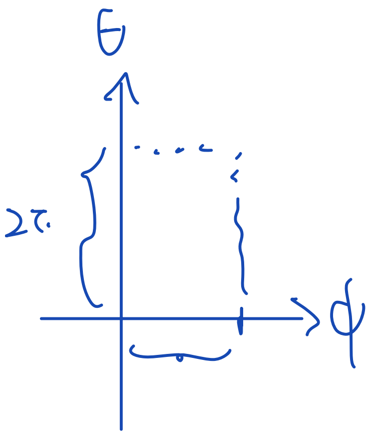
$$T_pS = \{\alpha \mathbf{x}_u + \beta \mathbf{x}_v : \alpha, \beta \in \mathbb{R}\} \subset \mathbb{R}^3.$$

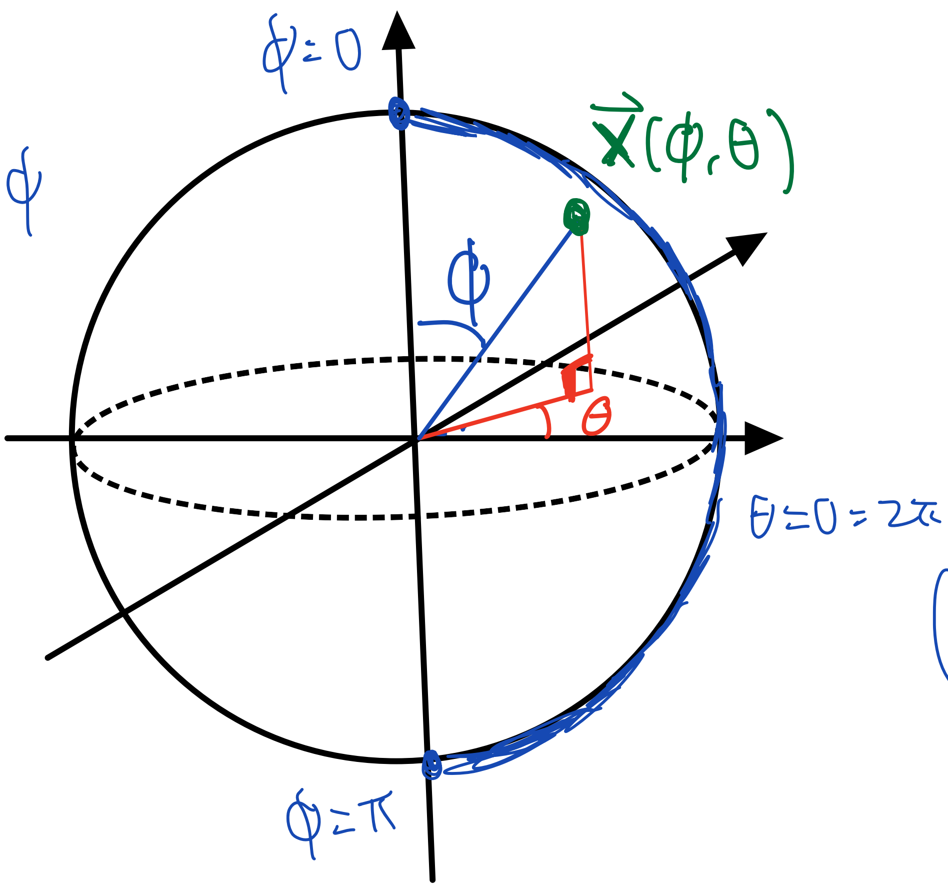


1. *Sphere: Let $r > 0$ be a positive real number. The function*

$$\mathbf{x}(\phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi), \text{ for } (\phi, \theta) \in (0, \pi) \times (0, 2\pi)$$

defines a **sphere** *of radius r centered at the origin.*





A, B set

$(a,b) \in A \times B$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

(x,y)

(r, θ, ϕ)

Spherical coordinate

2. *Torus: Let $R > r > 0$ be positive real numbers. The function*

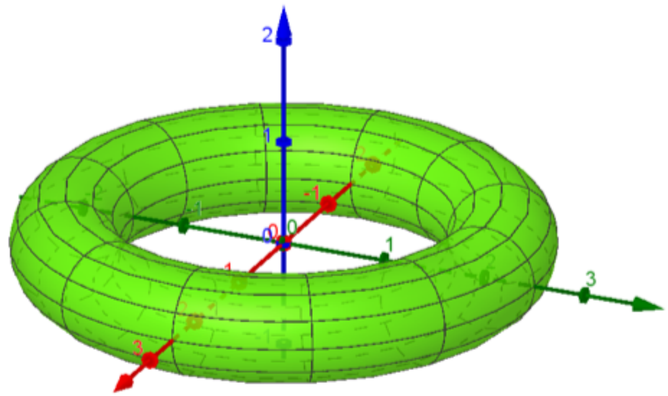
$$\mathbf{x}(\phi,\theta) = ((R+r\sin\phi)\cos\theta, (R+r\sin\phi)\sin\theta, r\cos\phi), \quad \textit{for } \phi,\theta \in (0,2\pi)$$

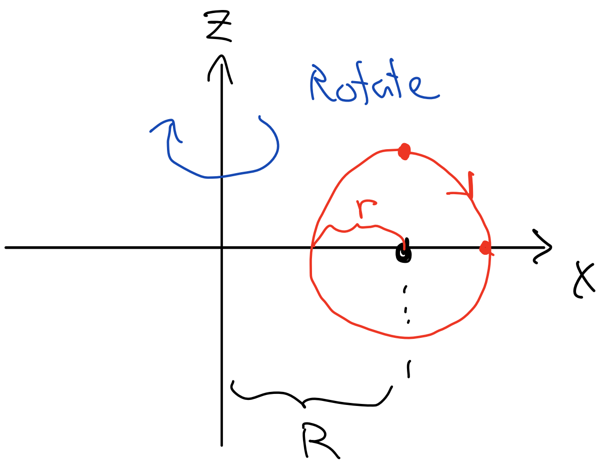
defines a regular surface which is called **torus**.

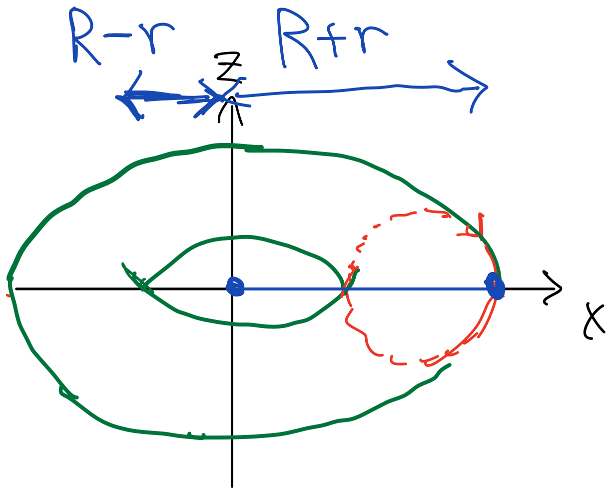
$R > r > 0$

if $\theta = 0$ $\quad X(\phi,0) = (R+r\sin\phi, 0, r\cos\phi)$

$$= (R,0,0) + \underline{r(\sin\phi, 0, \cos\phi)}$$







$\theta = 0$

$\phi \in (0, 2\pi)$

rotate around z-axis

$\theta \in (0, 2\pi)$

$\phi \in (0, 2\pi)$

3.2 First fundamental form and surface area

Definition 3.2.1 (First fundamental form). *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface. The* **first fundamental form** *of* $\mathbf{x}$ *is the* 2×2 *matrix valued function*

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix}.$$

Symmetric

Theorem 3.2.2. *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface and* I *be its first fundamental form. Then*

$$\det(I) = \|\mathbf{x}_u \times \mathbf{x}_v\|^2.$$

In particular, we have $\det(I) > 0$ *for any* u, v.

Pf

$$\begin{aligned} \det I &= \langle X_u, X_u \rangle \langle X_v, X_v \rangle - \langle X_u, X_v \rangle^2 \\ &= \|X_u\|^2 \|X_v\|^2 - (\|X_u\| \|X_v\| \cos\theta)^2 \\ &= \|X_u\|^2 \|X_v\|^2 (1 - \cos^2\theta) \\ &= \|X_u\|^2 \|X_v\|^2 \sin^2\theta = \|X_u \times X_v\|^2 \end{aligned}$$

Formula

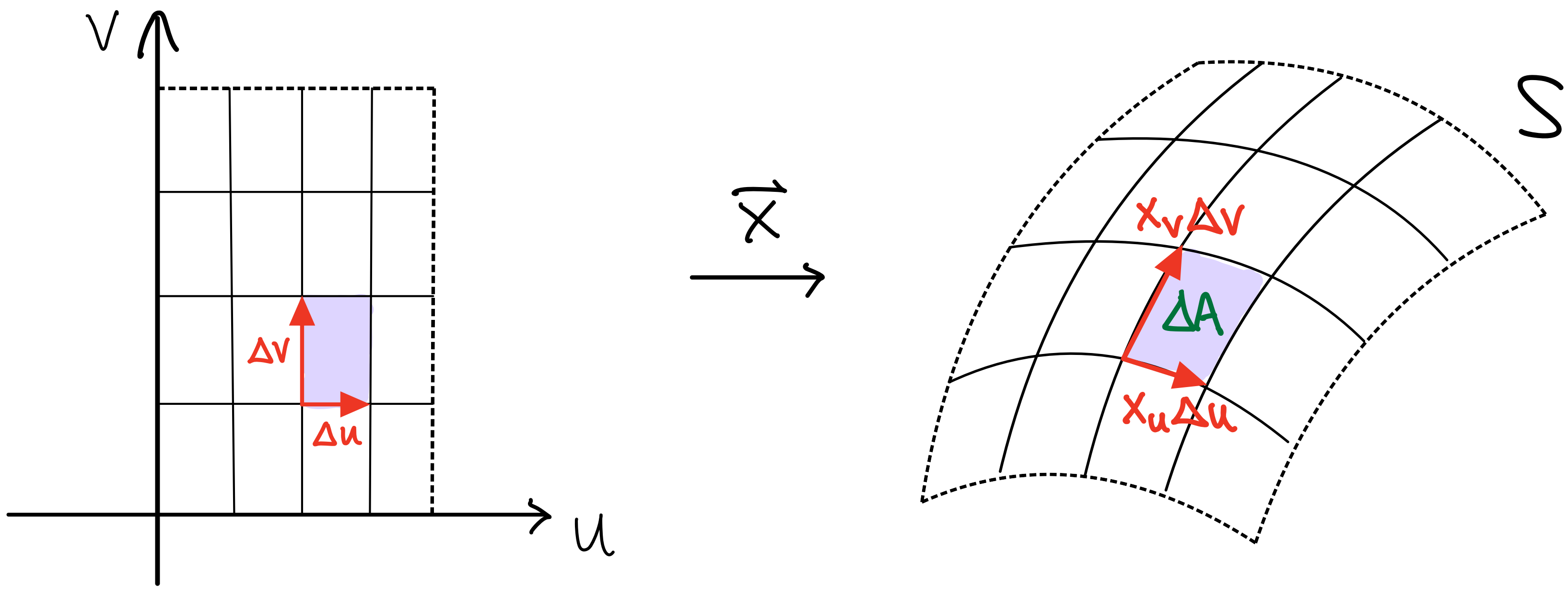
$$\langle a, b \rangle^2 + \|a \times b\|^2 = \|a\|^2 \|b\|^2$$

$$\|a \times b\| = \|a\| \|b\| \sin\theta$$

$$\langle a, b \rangle = \|a\| \|b\| \cos\theta$$

Definition 3.2.3 (Surface area)**.** *Let S be a regular surface with parametrization $\mathbf{x}(u,v)$, $(u,v) \in D$. The* **surface area** *of S is defined by*

$$A = \iint_D \sqrt{\det(I)}\, dudv.$$



$$\Delta A \approx \|\Delta u \mathbf{x}_u \times \Delta v \mathbf{x}_v\| = \|\mathbf{x}_u \times \mathbf{x}_v\| \Delta u \Delta v.$$

$\det I = \|x_u \times x_v\|^2$

$$A = \underbrace{\lim \sum}_{} \|\mathbf{x}_u \times \mathbf{x}_v\| \Delta u \Delta v = \iint_D \|\mathbf{x}_u \times \mathbf{x}_v\| dudv.$$

More accurate with more smaller parallelograms

Sphere: The function

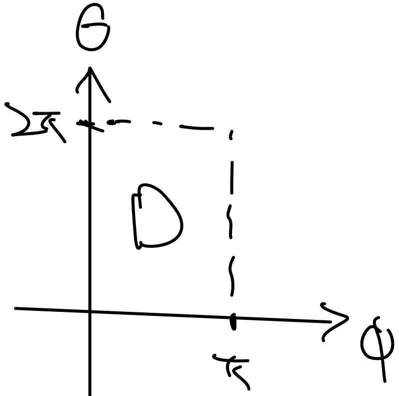
$$\mathbf{x}(\phi,\theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi), \; 0<\phi<\pi, 0<\theta<2\pi$$

$$X_\phi = (r\cos\phi\cos\theta, \; r\cos\phi\sin\theta, \; -r\sin\phi)$$

$$X_\theta = (-r\sin\phi\sin\theta, \; r\sin\phi\cos\theta, \; 0)$$

$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle X_u, X_u\rangle & \langle X_u, X_v\rangle \\ \langle X_v, X_u\rangle & \langle X_v, X_v\rangle \end{bmatrix}$$

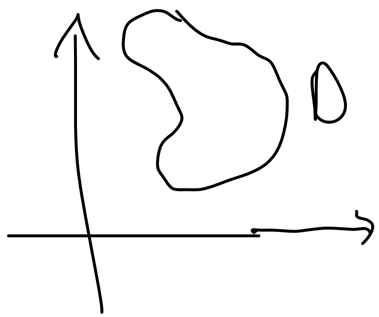
$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle X_\phi, X_\phi\rangle & \langle X_\phi, X_\theta\rangle \\ \langle X_\theta, X_\phi\rangle & \langle X_\theta, X_\theta\rangle \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2\sin^2\phi \end{bmatrix}$$



$$\sqrt{\det I} = \sqrt{r^4\sin^2\phi} = r^2\sin\phi$$

$$A = \iint_D \sqrt{\det I}\, d\phi\, d\theta = \int_0^{2\pi}\left(\int_0^{\pi}\sqrt{\det I}\, d\phi\right) d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{\pi} r^2\sin\phi\, d\phi\right) d\theta = \int_0^{2\pi}\left[-r^2\cos\phi\right]_0^{\pi} d\theta$$



$$= \int_0^{2\pi} 2r^2\, d\theta = \left[2r^2\theta\right]_0^{2\pi} = 4\pi r^2$$

2. *Torus: The function*

$$\mathbf{x}(\phi, \theta) = ((R + r\sin\phi)\cos\theta, (R + r\sin\phi)\sin\theta, r\cos\phi), 0 < \phi, \theta < 2\pi$$

$$X_\phi = (r\cos\phi\cos\theta, \ r\cos\phi\sin\theta, \ -r\sin\phi)$$

$$X_\theta = ((R+r\sin\phi)(-\sin\theta), \ (R+r\sin\phi)\cos\theta, \ 0)$$

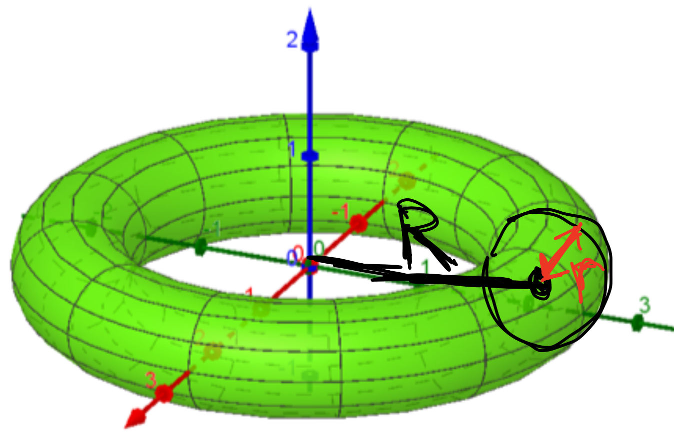
$$I = \begin{bmatrix} \langle X_\phi, X_\phi\rangle & \langle X_\phi, X_\theta\rangle \\ \langle X_\theta, X_\phi\rangle & \langle X_\theta, X_\theta\rangle \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & (R+r\sin\phi)^2 \end{bmatrix}$$

$$\sqrt{\det I} = r(R+r\sin\phi)$$

$$A = \int_0^{2\pi}\int_0^{2\pi}\sqrt{\det I}\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{2\pi} r(R+r\sin\phi)\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\left[r(R+r\sin\phi)\theta\right]_0^{2\pi} d\phi = \int_0^{2\pi} r(R+r\sin\phi)2\pi\, d\phi$$

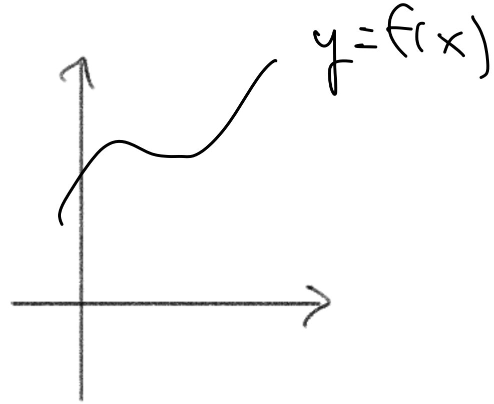
$$= 4\pi^2 rR$$



Theorem 3.2.5 (Surface area of graphs of functions).

1. **Rectangular coordinates**: *Let $z = f(x,y)$, $(x,y) \in D \subset \mathbb{R}^2$, be a differentiable function. The surface area of the graph of $z = f(x,y)$ in rectangular coordinates is*

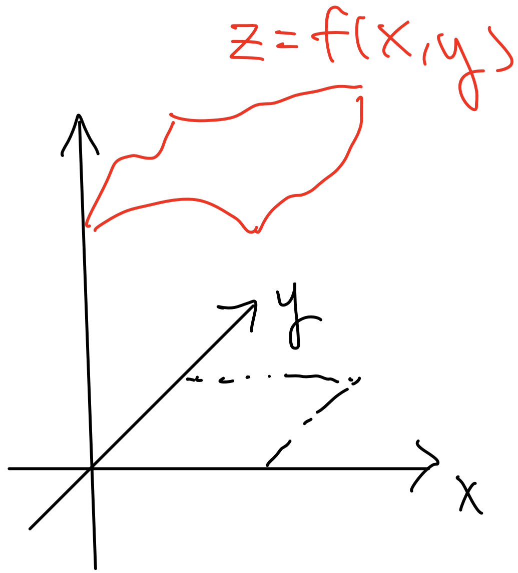
$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2}\,dxdy.$$



Parametrization: $\vec{X} = (x, y, f(x,y))$

$\vec{X}_x = (1, 0, f_x)$ $\quad$ $\vec{X}_y = (0, 1, f_y)$

$$I = \begin{bmatrix} 1+f_x^2 & f_xf_y \\ f_xf_y & 1+f_y^2 \end{bmatrix}$$



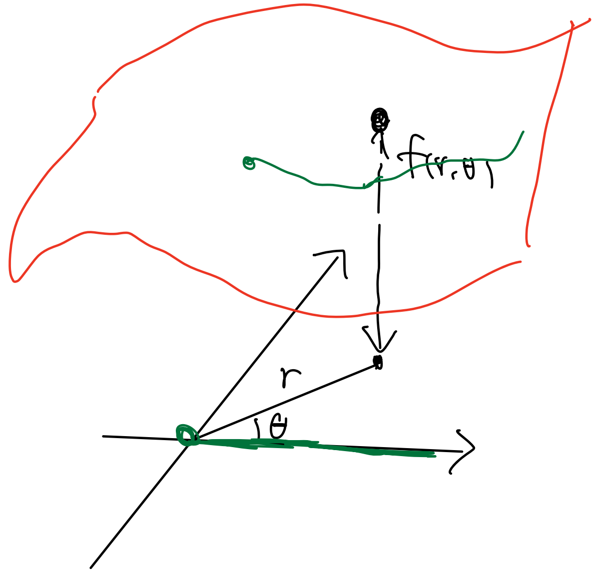
$$\det I = (1+f_x^2)(1+f_y^2) - f_x^2f_y^2$$

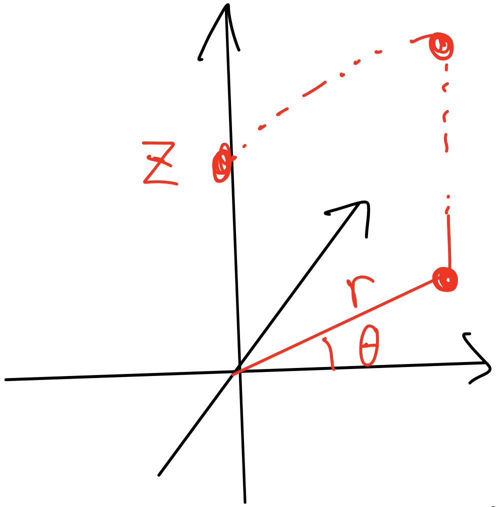
$$= 1 + f_x^2 + f_y^2$$

2. **Cylindrical coordinates**: *Let $z = f(r,\theta)$, $(r,\theta) \in D \subset \mathbb{R}^+ \times (0, 2\pi)$, be a differentiable function. The surface area of the graph of $z = f(r,\theta)$ in cylindrical coordinates is*

$$A = \iint_D \sqrt{r^2 + r^2 f_r^2 + f_\theta^2}\, dr\, d\theta.$$

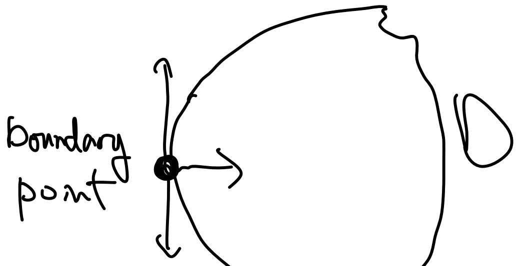
$$\vec{X}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ f(r,\theta))$$





Cylindrical coordinates

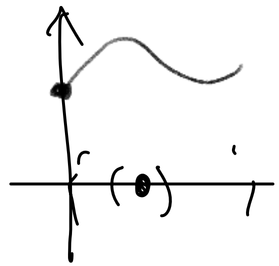
r, θ, z



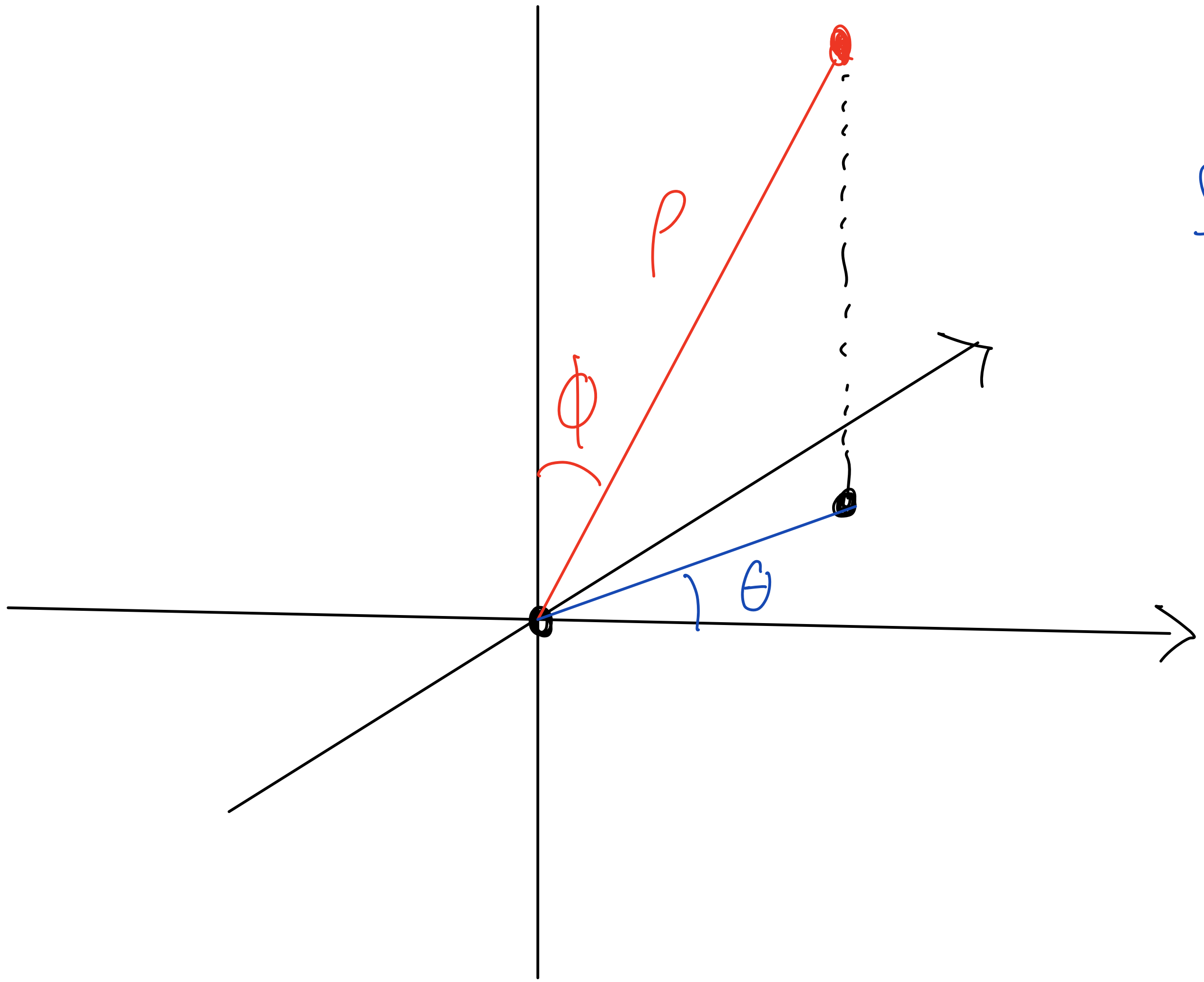
D open is better for differentiation

$f : [0,1] \to \mathbb{R}$

$f : (0,1) \to \mathbb{R}$



Space

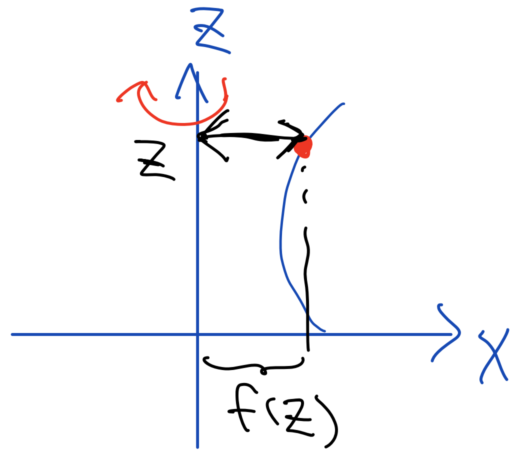


Spherical Coordinates

(ρ, θ, ϕ)

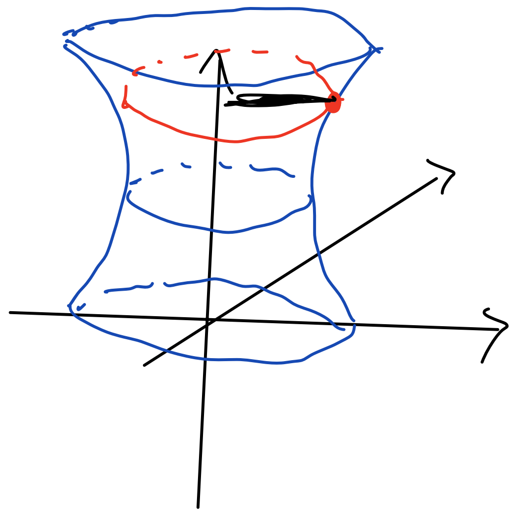
Theorem 3.2.6 (Surface area of surfaces of revolution). *Let $f(z) > 0$, $z \in (a, b)$ be a positive differentiable function. The surface area of the surface obtained by rotating the graph of $x = f(z)$ in the xz-plane about the z axis is*

$$A = 2\pi \int_a^b f\sqrt{1 + f'^2}\,dz.$$



Parametrization

$$\vec{X} = (f(Z)\cos\theta, f(Z)\sin\theta, Z)$$



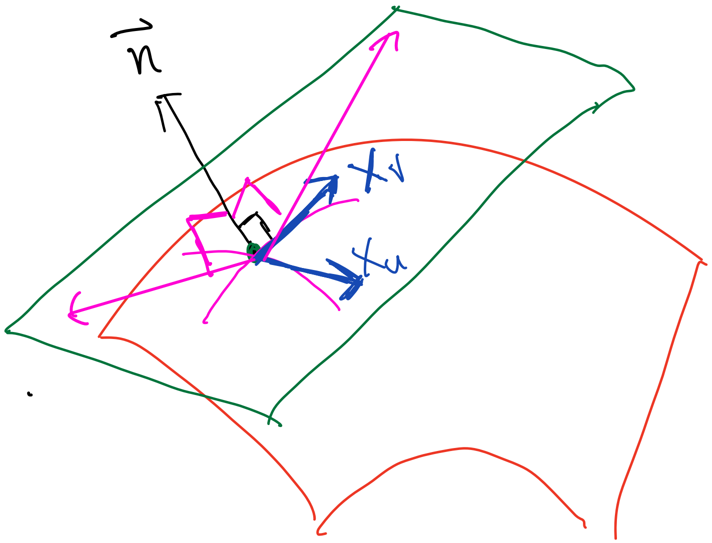
$\boxed{12:00}$

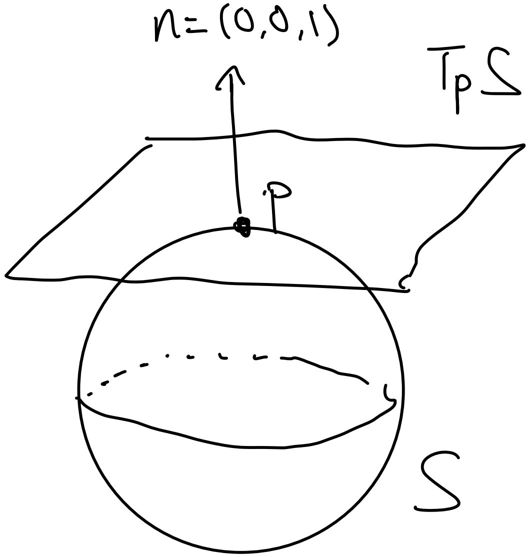
3.3 Second fundamental form and Gaussian curvature

Definition 3.3.1 (Unit normal vector). *Let* $\mathbf{x}(u, v)$ *be a regular parametrized surface. The* **unit normal vector** *to the surface is*

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

$X_u \times X_v = - X_v \times X_u$





$T_pS = \{\alpha(1,0,0) + \beta(0,0,1) : \alpha, \beta \in \mathbb{R}\}$

$= \{v \in \mathbb{R}^3 : \langle v, (0,0,1)\rangle = 0\}$

Proposition 3.3.2. *Let* S *be a regular surface with parametrization* $\mathbf{x}(u, v)$. *Let* T_pS *be the tangent space to the surface at a point* $p = \mathbf{x}(u, v)$. *Then*

$$T_pS = \{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v}, \mathbf{n}\rangle = 0\}.$$

Definition 3.3.3 (Second fundamental form). *Let* $\mathbf{x}(u,v)$ *be a regular parametrized surface which has continuous second derivatives. The* **second fundamental form** *is the* 2×2 *matrix valued function*

$$II = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{n}\rangle & \langle \mathbf{x}_{uv}, \mathbf{n}\rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n}\rangle & \langle \mathbf{x}_{vv}, \mathbf{n}\rangle \end{pmatrix} = -\begin{pmatrix} \langle \mathbf{x}_{u}, \mathbf{n}_u\rangle & \langle \mathbf{x}_{u}, \mathbf{n}_v\rangle \\ \langle \mathbf{x}_{v}, \mathbf{n}_u\rangle & \langle \mathbf{x}_{v}, \mathbf{n}_v\rangle \end{pmatrix}.$$

Rmk ① $X_{uv} = (X_u)_v = (X_v)_u \Rightarrow II$ symmetric

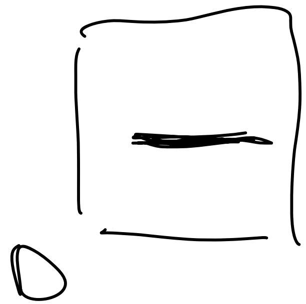
② $\vec{n} = \dfrac{X_u \times X_v}{\|X_u \times X_v\|} \perp X_u$

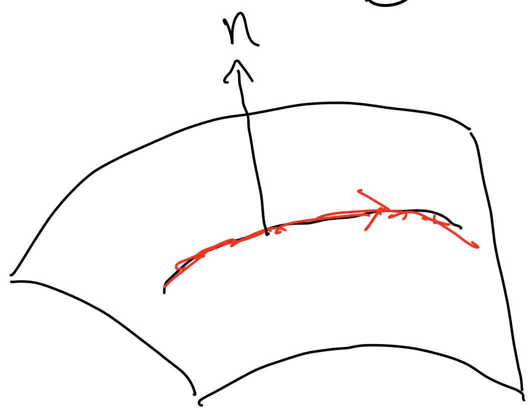
$\Rightarrow \langle X_u, n\rangle \equiv 0$

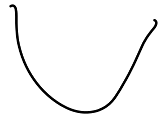
$\Rightarrow \langle X_u, n\rangle_u \equiv 0$

$\Rightarrow \langle X_{uu}, n\rangle + \langle X_u, n_u\rangle \equiv 0$

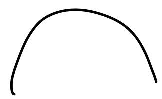
$\langle X_{uu}, n\rangle = -\langle X_u, n_u\rangle$







$f'' > 0$

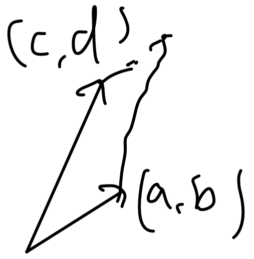


$f'' < 0$

Definition 3.3.4 (Gaussian curvature). *Let* $\mathbf{x}(u, v)$ *be a regular parametrized surface which has continuous second derivatives. The* **Gaussian curvature** *of the surface is*

$$K = \frac{\det(II)}{\det(I)} = \frac{eg - f^2}{EG - F^2}$$

where I *is the first fundamental form and* II *is the second fundamental form of the surface.*



$A = |ad - bc|$

<u>Rmk</u>

$$K = \frac{\begin{vmatrix} \langle X_{uu}, n\rangle & \langle X_{uv}, n\rangle \\ \langle X_{vu}, n\rangle & \langle X_{vv}, n\rangle \end{vmatrix}}{\| X_u \times X_v \|^2}$$

$$\det\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \| X_u \times X_v \|^2$$

Curve analogue

$$\kappa = \frac{\langle r'', N\rangle}{\|r'\|^2}$$

$$r'' = \frac{dv}{dt}\mathbf{T} + \kappa v^2 N$$

$$\langle r'', N\rangle = \langle \kappa v^2 N, N\rangle$$
$$= \kappa v^2$$

Example 3.3.5.

1. *Sphere: A sphere of radius r centered at the origin is parametrized by*

$$\mathbf{x}(\phi,\theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi),\ \ 0<\phi<\pi, 0<\theta<2\pi.$$

$$\begin{cases} \mathbf{x}_\phi = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi) \\ \mathbf{x}_\theta = (-r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0). \end{cases}$$

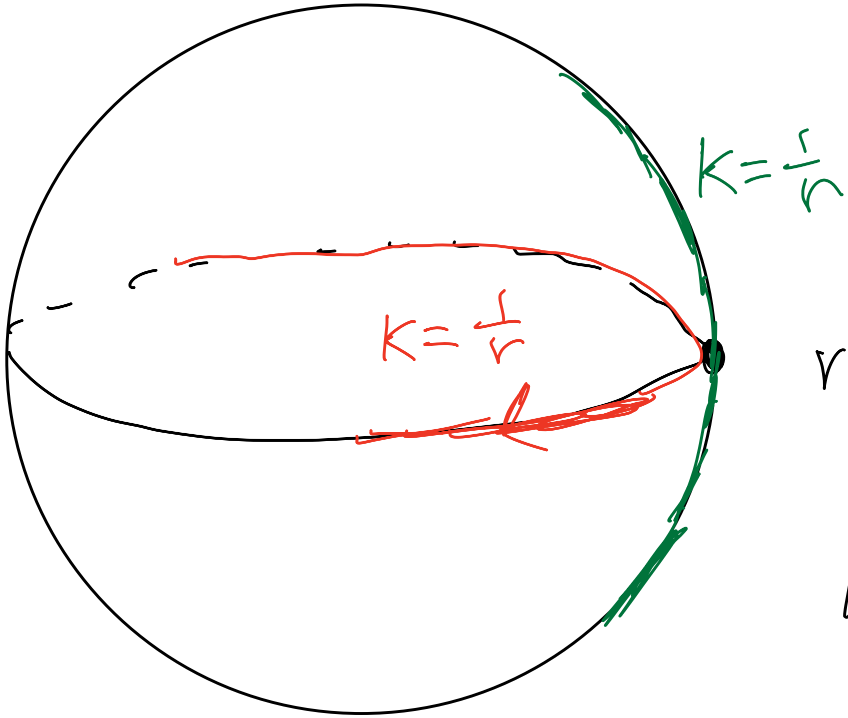
$$\begin{aligned} I &= \begin{pmatrix} \langle \mathbf{x}_\phi, \mathbf{x}_\phi\rangle & \langle \mathbf{x}_\phi, \mathbf{x}_\theta\rangle \\ \langle \mathbf{x}_\theta, \mathbf{x}_\phi\rangle & \langle \mathbf{x}_\theta, \mathbf{x}_\theta\rangle \end{pmatrix} \\ &= \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\phi \end{pmatrix}. \end{aligned}$$

$$\mathrm{II} = -\begin{bmatrix} \langle X_\phi, n_\phi\rangle & \langle X_\phi, n_\theta\rangle \\ \langle X_\theta, n_\phi\rangle & \langle X_\theta, n_\theta\rangle \end{bmatrix} = \begin{bmatrix} -r & 0 \\ 0 & -r\sin^2\phi \end{bmatrix}$$

$$X_\phi \times X_\theta = (r^2\sin^2\phi\cos\theta, r^2\sin^2\phi\sin\theta, r^2\sin\phi\cos\phi)$$

$$\vec{n} = \frac{X_\phi \times X_\theta}{\underbrace{\|X_\phi\times X_\theta\|}_{r^2\sin\phi}} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$K = \frac{\det \mathrm{II}}{\det I} = \frac{r^2\sin^2\phi}{r^4\sin^2\phi} = \frac{1}{r^2}$$



radius r

$$K = \underbrace{\frac{1}{r} \frac{1}{r}} = \frac{1}{r^2}$$

Just this example

Not always true

2. *Torus: Let $R > r > 0$ be constants. The function*

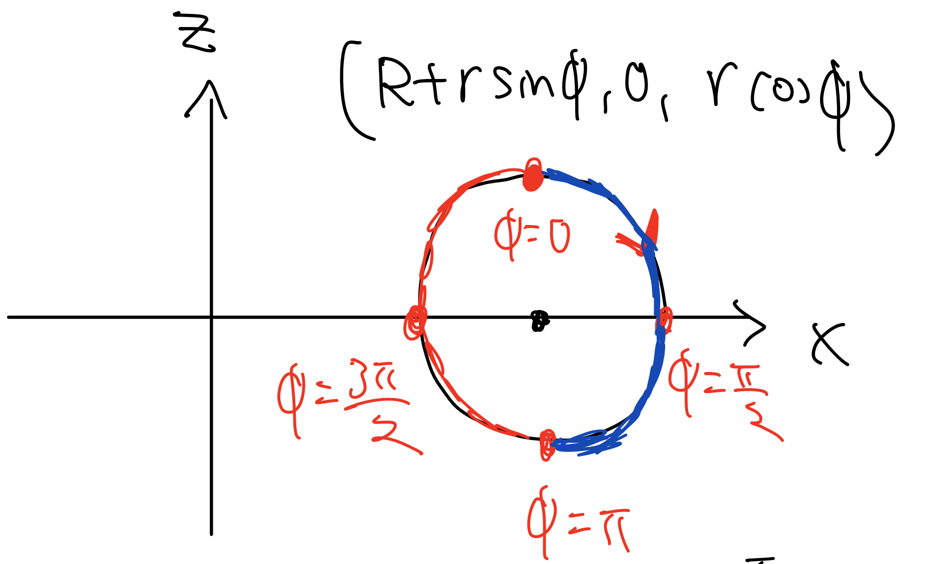
$$\mathbf{x}(\phi,\theta) = ((R + r\sin\phi)\cos\theta, (R + r\sin\phi)\sin\theta, r\cos\phi),\ \ 0 < \phi, \theta < 2\pi$$

$\theta = 0$

$$\begin{cases} \mathbf{x}_\phi = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi) \\ \mathbf{x}_\theta = (-(R + r\sin\phi)\sin\theta, (R + r\sin\phi)\cos\theta, 0). \end{cases}$$

The first fundamental form is

$$I = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\sin\phi)^2 \end{pmatrix}.$$

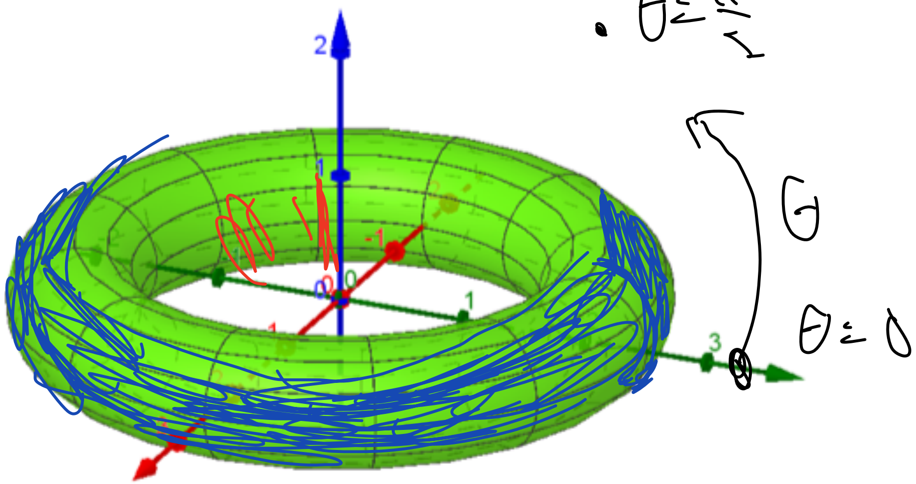


$$K = \frac{\sin\phi}{r(R + r\sin\phi)} > 0$$

$\frac{\pi}{2}$

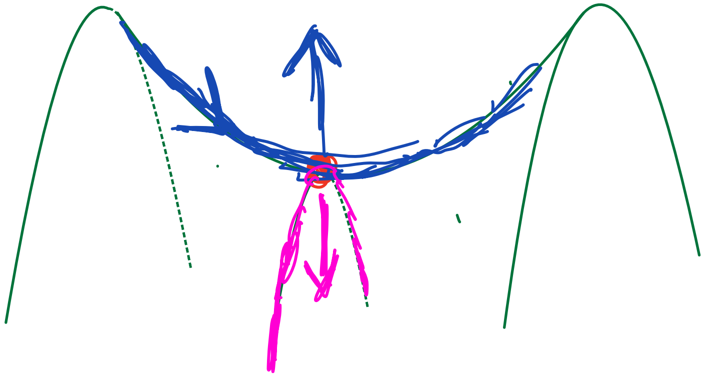
$$\begin{cases} \geq 0 & \text{if} \quad \phi \in [0,\pi] \\ \leq 0 & \text{if} \quad \phi \in [\pi, 2\pi] \end{cases}$$

$\theta = \frac{\pi}{2}$

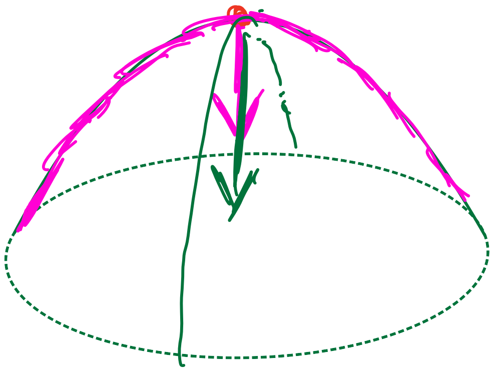


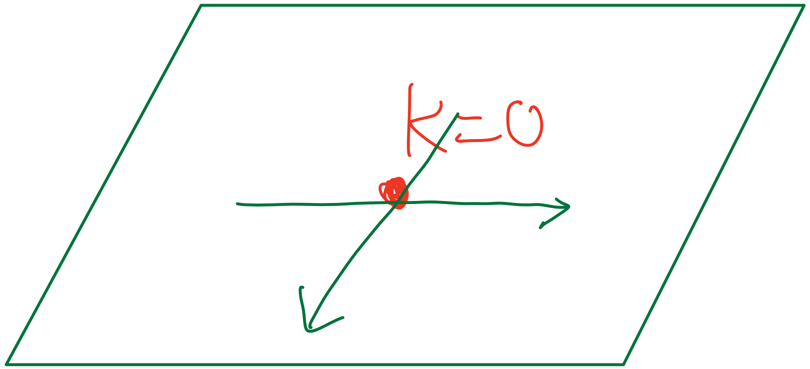
$\frac{3\pi}{2}$

$K<0$ Saddle

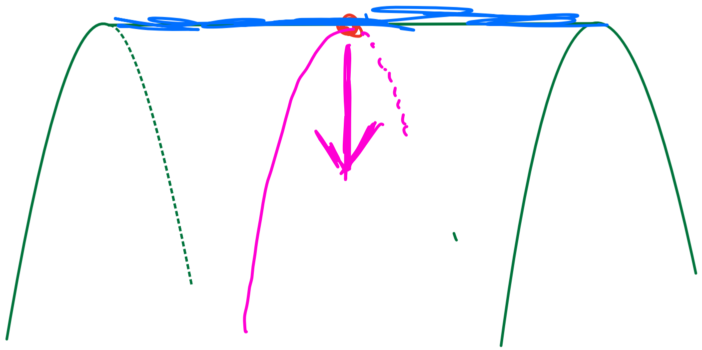


$K>0$





$K=0$



Proposition 3.3.6 (Curvature of graphs of functions).

1. *Let $f(x,y)$, $(x,y) \in D \subset \mathbb{R}^2$, be a function with continuous second derivatives. The Gaussian curvature of the graph of $z = f(x,y)$ in rectangular coordinates is*

$$K(x,y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

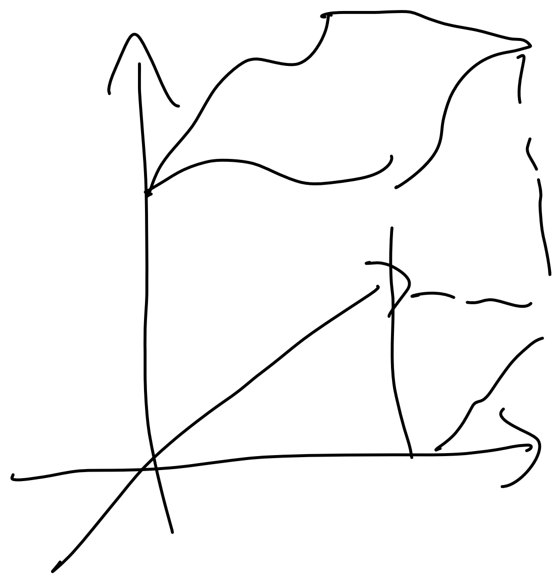
Proof. 1. The surface is parametrized by $\mathbf{x}(x,y) = (x, y, f(x,y))$, $(x,y) \in D$. Then

$$\begin{cases} \mathbf{x}_x = (1, 0, f_x) \\ \mathbf{x}_y = (0, 1, f_y) \end{cases}$$

and the first fundamental form is

$$\begin{aligned} I &= \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} \\ &= \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}. \end{aligned}$$

$z = f(x,y)$



Exercise

2. *Let* $f(r,\theta)$, $(r,\theta) \in D \subset \mathbb{R}^+ \times (0, 2\pi)$, *be a function with continuous second derivatives. The Gaussian curvature of the graph of* $z = f(r,\theta)$ *in cylindrical coordinates is*

$$K(r,\theta) = \frac{r^2 f_{rr}(r f_r + f_{\theta\theta}) - (r f_{r\theta} - f_\theta)^2}{(r^2 + r^2 f_r^2 + f_\theta^2)^2}.$$

Exercise

Proposition 3.3.7 (Gaussian curvature of surfaces of revolution).

1. **By graph of function**: *Let $f(z)$, $z \in (a,b)$, be a function with continuous second derivative. The Gaussian curvature of the surface obtained by rotating the graph of* $\boxed{x = f(z)}$ *on the xz-plane about the z axis is*

$\varphi = f(z)$

$\psi = z$

$$K(z) = -\frac{f''}{f(1+f'^2)^2}.$$

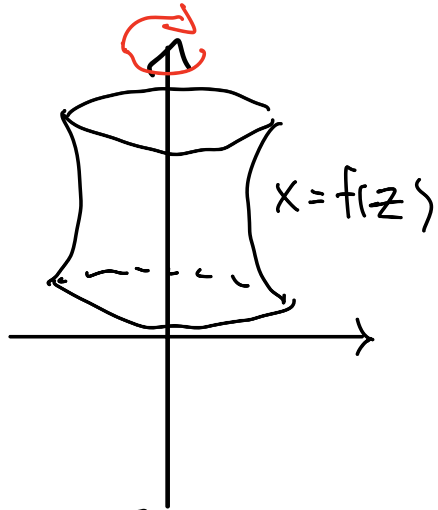
$(f(z), z)$

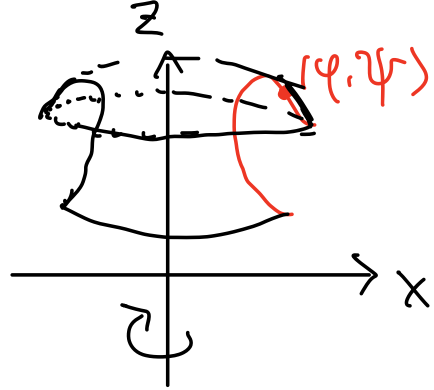
2. **By parametrized curve**: *Let $(\varphi(u), \psi(u))$, $u \in (a,b)$, be a regular parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(u), \psi(u))$ on the xz-plane about the z axis is*

$$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}.$$

3. **By arc length parametrized curve**: *Let $(\varphi(s), \psi(s))$, $s \in (a,b)$, be an arc length parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(s), \psi(s))$ on the xz-plane about the z axis is*

$$K(s) = -\frac{\varphi''}{\varphi}.$$





② is the strongest version and implies ①, ③

2. **By parametrized curve**: *Let $(\varphi(u), \psi(u))$, $u \in (a, b)$, be a regular parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x, z) = (\varphi(u), \psi(u))$ on the xz-plane about the z axis is*

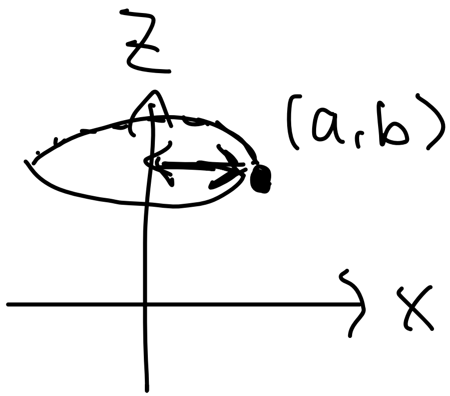
$$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}.$$

<u>Pf</u> Parametrize the surface of revolution by parametrized curve $(x, z) = (\varphi(u), \psi(u))$ by

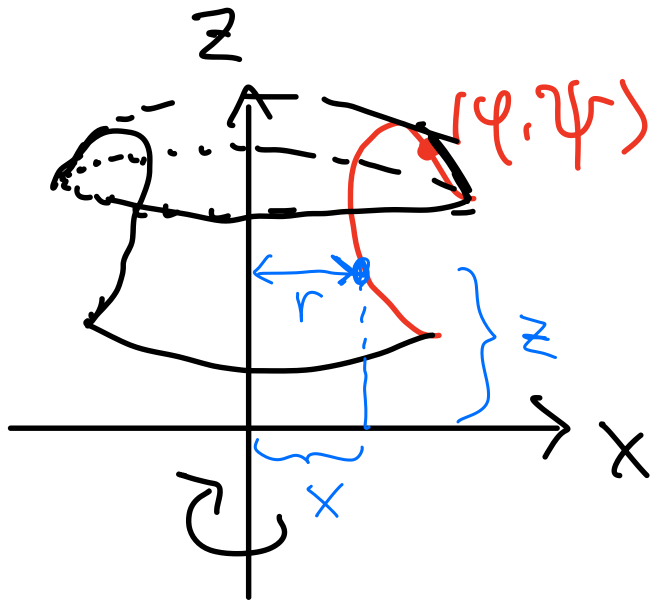
$$\mathbf{x}(u, \theta) = (\underbrace{\varphi(u)}_{r} \cos\theta, \underbrace{\varphi(u)}_{r} \sin\theta, \psi(u)).$$

ψ = psi

φ = phi



$(a\cos\theta, a\sin\theta, b)$



2. **By parametrized curve**: *Let $(\varphi(u), \psi(u))$, $u \in (a, b)$, be a regular parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x, z) = (\varphi(u), \psi(u))$ on the xz-plane about the z axis is*

$$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}.$$

Parametrize the surface of revolution by parametrized curve $(x, z) = (\varphi(u), \psi(u))$ by

$$\mathbf{x}(u, \theta) = (\varphi(u)\cos\theta, \varphi(u)\sin\theta, \psi(u)).$$

$$X_u = (\varphi'\cos\theta, \varphi'\sin\theta, \psi') \qquad X_\theta = (-\varphi\sin\theta, \varphi\cos\theta, 0)$$

$$X_{uu} = (\varphi''\cos\theta, \varphi''\sin\theta, \psi'') \qquad X_{\theta\theta} = (-\varphi\cos\theta, -\varphi\sin\theta, 0)$$

$$X_{u\theta} = (-\varphi'\sin\theta, \varphi'\cos\theta, 0)$$

$$X_u \times X_\theta = \begin{vmatrix} i & j & k \\ \varphi'\cos\theta & \varphi'\sin\theta & \psi' \\ -\varphi\sin\theta & \varphi\cos\theta & 0 \end{vmatrix} = (-\varphi\psi'\cos\theta, -\varphi\psi'\sin\theta, \varphi\varphi')$$

$$\|X_u \times X_\theta\| = \varphi\sqrt{\varphi'^2 + \psi'^2} \qquad (\text{assume } \varphi > 0)$$

$$\vec{n} = \frac{X_u \times X_\theta}{\|X_u \times X_\theta\|} = \frac{1}{\sqrt{\varphi'^2 + \psi'^2}}(-\psi'\cos\theta, -\psi'\sin\theta, \varphi')$$

$$\mathrm{I} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \varphi'^2 + \psi'^2 & 0 \\ 0 & \varphi^2 \end{bmatrix}$$

$$\mathrm{II} = \begin{bmatrix} e & f \\ f & g \end{bmatrix} = \frac{1}{\sqrt{\varphi'^2 + \psi'^2}} \begin{bmatrix} -\varphi''\psi' + \varphi'\psi'' & 0 \\ 0 & \varphi\psi' \end{bmatrix}$$

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \frac{1}{\varphi'^2 + \psi'^2} (\varphi'\psi'' - \varphi''\psi')\varphi\psi' \cdot \frac{1}{\varphi^2(\varphi'^2 + \psi'^2)}$$

$$= \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}$$

$$\det \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$$

$$= 2 \det \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

$$= 2^2 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Proposition 3.3.7 (Gaussian curvature of surfaces of revolution).

1. **By graph of function**: *Let $f(z)$, $z \in (a,b)$, be a function with continuous second derivative. The Gaussian curvature of the surface obtained by rotating the graph of $x = f(z)$ on the xz-plane about the z axis is*

$$K(z) = -\frac{f''}{f(1+f'^2)^2}.$$

2. **By parametrized curve**: *Let $(\varphi(u), \psi(u))$, $u \in (a,b)$, be a regular parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(u), \psi(u))$ on the xz-plane about the z axis is*

$$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)^2}.$$

3. **By arc length parametrized curve**: *Let $(\varphi(s), \psi(s))$, $s \in (a,b)$, be an arc length parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(s), \psi(s))$ on the xz-plane about the z axis is*

$$K(s) = -\frac{\varphi''}{\varphi}.$$

② ⇒ ① Put $(\varphi, \psi) = (f(z), z)$. Then

$$K = \frac{z'(f'z'' - f''z')}{f(f'^2 + z'^2)^2} = \frac{-f''}{f(1+f'^2)^2}$$

② ⇒ ③

If $\varphi'^2 + \psi'^2 = 1$, then

$$K = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi}$$

$$= \frac{\varphi'\psi'\psi'' - \varphi''\psi'^2}{\varphi}$$

$$\overset{*}{=} \frac{-\varphi'\varphi'\varphi'' - \varphi''\psi'^2}{\varphi}$$

$$= \frac{-\varphi''(\varphi'^2 + \psi'^2)}{\varphi} = -\frac{\varphi''}{\varphi}$$

* $\varphi'^2 + \psi'^2 \equiv 1$

$\Rightarrow 2\varphi'\varphi'' + 2\psi'\psi'' = 0$

$\psi'\psi'' = -\varphi'\varphi''$